Worst-Case Optimal Graph Joins in Almost No Space

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Abstract

We present an indexing scheme that supports worst-case optimal (wco) joins over graphs within compact space. Supporting all possible wco joins using conventional data structures – based on B(+)-Trees, tries, etc. – requires 6 index orders in the case of graphs represented as triples. We rather propose a form of index, which we call a *ring*, that indexes each triple as a set of cyclic bidirectional strings of length 3. Rather than maintaining 6 orderings, we can use one ring to index them all. This ring replaces the graph and uses only sublinear extra space on top of the graph; in order words, the ring supports worst-case optimal graph joins in almost no space beyond storing the graph itself. We perform experiments using our representation to index a large graph (Wikidata) in memory, over which wco join algorithms are implemented. Our experiments show that the ring offers the best overall performance for query times while using only a small fraction of the space when compared with several state-of-the-art approaches.

CCS Concepts

• **Theory of computation → Database query processing and optimization (theory)**; **Data structures and algorithms for data management**.

Keywords

Worst-case optimal joins; graph patterns; graph databases; graph indexing; Burrows–Wheeler transform; wavelet trees

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R

x	y
1	2
1	3
2	3

S

y	z
2	4
3	4
3	5

T

x	z
1	4
2	3
3	2

R ⋈ S ⋈ T

x	y	z
1	2	4
1	3	4

Figure 1: Example of three relations and their natural join

1 Introduction

A recent development in the efficient processing of database queries is that of *worst-case optimal* (*wco*) join algorithms, which process join queries in time proportional to the *AGM bound* [6]: the maximum possible output size produced by a join query over a relational database of a certain size. Such algorithms can be strictly better than any traditional query plan using pairwise joins [38, 44].

Leapfrog-TrieJoin (LTJ) is a seminal wco join algorithm based on "eliminating" attributes from a query [44]. To illustrate the algorithm (defined in more detail in Section 2.2), consider the relations R, S, T in Figure 1 along with the query $Q = \mathsf{R} \bowtie \mathsf{S} \bowtie \mathsf{T}$ computing their natural join. In order to evaluate this query, LTJ assumes an ordering of the attributes in Q, say (x, y, z) (details of this ordering are discussed later). For the first attribute x, LTJ finds all constants a such that the query $\sigma_{\mathsf{x}=a}(\mathsf{R} \bowtie \mathsf{T})$ gives some solution, here joining all relations that mention x in the join (R and T). In this case $\sigma_{\mathsf{x}=1}(\mathsf{R} \bowtie \mathsf{T})$ and $\sigma_{\mathsf{x}=2}(\mathsf{R} \bowtie \mathsf{T})$ give solutions, while $\sigma_{\mathsf{x}=3}(\mathsf{R} \bowtie \mathsf{T})$ does not. We thus say that 1 and 2 eliminate x. Next LTJ eliminates y: for each constant a found to eliminate x in the previous step, we find all constants b that eliminate y, that is, such that $\sigma_{\mathsf{x}=a \wedge \mathsf{y}=b}(\mathsf{R} \bowtie \mathsf{S})$ gives solutions. Given $a = 1$, we find $b = 2$ and $b = 3$, while given $a = 2$ we find $b = 3$. We thus say that $(1, 2)$, $(1, 3)$ and $(2, 3)$ eliminate (x, y). Finally LTJ eliminates z: for each elimination (a, b) of (x, y) computed previously, we find all constants c that eliminate z. Given $(a, b) = (1, 2)$, we find $c = 4$; given $(a, b) = (1, 3)$, we again find $c = 4$; given $(a, b) = (2, 3)$ we find no valid eliminations. Since the tuples $(1, 2, 4)$ and $(1, 3, 4)$ eliminate all attributes (x, y, z), they are thus the final solutions of the query Q computed by LTJ.

In order for LTJ to satisfy wco guarantees, the constants that eliminate a given attribute must be enumerated in time proportional to the number of results. This is typically enabled by indexing each relation such that we can assign constants to any subset of attributes

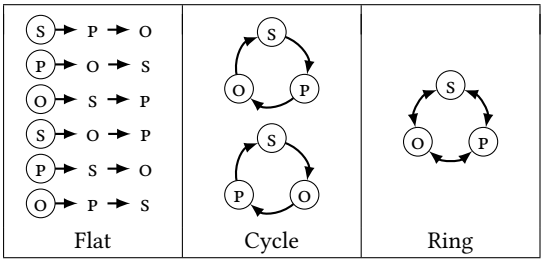


Figure 2: Illustration of triple indexing schemes

of the relation and efficiently enumerate the results for any other attribute in sorted order, which enables the efficient intersection of results coming from multiple relations for that attribute. However, if using traditional data structures with prefix lookups (B(+)Trees, tries, etc.), the number of index orders required is exponential in the arity of the relation. Veldhuizen [44] thus proposes to load indexes lazily, which may incur high runtime costs and space overhead.

Various works have found that wco join algorithms are well-suited for graphs, where graphs have low fixed arity, and graph queries (thus) tend to feature many joins [1, 23, 24, 40]. But considering graphs with an arity of 3, a complete index supporting wco joins still requires 6 index orders, leading to high levels of redundancy and high space requirements; the aforementioned approaches have mostly focused on time rather than space in wco algorithms. Conversely, various proposals of compact data structures for graphs have been proposed [10, 11, 16, 46], with applications for archiving, publishing, in-memory databases and caching, etc. These proposals do not directly address wco joins. However, within the cyclic indexing scheme of Brisaboa et al. [10] we find the seeds of a general technique to reduce the space needed for evaluating wco joins.

Contribution. We propose an indexing scheme that greatly reduces space requirements for supporting wco joins. The scheme indexes graphs composed of a set of subject–predicate–object triples of the form (s, p, o) (each denotes an edge $s \xrightarrow{p} o$). Like the proposal of Brisaboa et al. [10], triples are indexed as cyclic strings of length 3. Unlike their proposal, we use text indexing techniques that consider triples as *bidirectional* cyclic strings, allowing us to support wco joins with only one index order, and thus with sublinear space beyond storing the graph itself. We call our indexing scheme a *ring*.

Figure 2 illustrates the indexing schemes, where we view a graph as a relation with attributes S, P, O, and circle attributes from which the index order starts. In the (traditional) flat indexing scheme, we require six orders for wco joins using LTJ, specifying constants for attributes in sequence of the given order; we can then read the first unbound attribute in sorted order. In the cyclic indexing scheme of Brisaboa et al. [10], we can start at any attribute, and proceed in the order shown, reading the first unbound attribute; in this case, two orders are needed to cover all patterns. Our proposal is the ring scheme, where we can traverse the attributes in either direction, allowing us to support wco graph joins with only one order.

We show how to implement this ring index using techniques inspired by text indexing, and characterise the space it requires. We prove that this implementation enables the evaluation of wco joins over graphs using an LTJ-style algorithm. Our experiments on the Wikidata graph show that the ring index uses 6% additional space beyond the raw data and 4–11 times less space than various prominent alternatives (Jena, RDF-3X, Blazegraph, Virtuoso) while being 2–36 times faster in general to solve basic graph patterns. An exception is EmptyHeaded [1], where our index is 140 times smaller and slightly faster overall. Only Qdag [35], a recent succinct index, is smaller than our basic ring index, but a compressed ring we develop is 150 times faster than Qdag while using 75% of the space it uses. We further discuss how rings could be applied in future work to reduce the indexes needed for wco joins in relational settings (e.g., for arity 6, we require 7 rings instead of 6! = 720 flat indexes).

Limitations. Our proposed indexing scheme, though efficient in space, is based on an in-memory data-structure that relies heavily on random accesses, which makes it difficult to migrate to disk and may be slower than in-memory indexes that enable more sequential access. The scheme is currently read-only, and uses relatively simple data loading and query planning techniques; we discuss in the conclusions how such issues can be concretely addressed as part of future work. While a ring in one order is sufficient for graphs, multiple orders are required for higher-arity relations (though far fewer than traditional indexing schemes, as discussed later). We currently focus on evaluating basic graph patterns; though support for other features of graph query languages could be simply layered on top, it may be possible in the future to optimise such features by pushing them to lower-level operations over the index.

2 Related works and concepts

We now introduce key concepts and works relating to graph joins and wco join algorithms, as well as text-indexing techniques.

2.1 Graph joins and patterns

2.1.1 Graphs. We adopt a relational view of graphs, treating them as a single ternary relation G. A triple (s, p, o) in the graph encodes an edge $s \xrightarrow{p} o$ from node s to node o labelled p, as seen in Figure 3. We assume the domain of graphs to be drawn from a totally ordered, countably infinite set $\mathcal{U}$ of constants. Given a constant $u \in \mathcal{U}$, we denote by $u + 1$ the next element in the total order after u in $\mathcal{U}$. We denote by $\mathrm{dom}(G)$ the set of all constants in G.

2.1.2 Graph patterns. A *basic graph pattern* is a graph in which some constants may be replaced by variables that can be matched against another graph. To be more precise, let $\mathcal{V}$ be an infinite set of variables, disjoint from $\mathcal{U}$. A *triple pattern* is then a tuple $(s, p, o) \in (\mathcal{U} \cup \mathcal{V})^3$, and a *basic graph pattern* is a set $Q \subseteq 2^{(\mathcal{U} \cup \mathcal{V})^3}$ of triple patterns. Each triple pattern represents an atomic query over the graph, and thus a basic graph pattern corresponds to a conjunctive query (aka. *join query*) over the relational representation of the graph. Let $\mathrm{vars}(Q)$ denote the set of variables used in Q. The *evaluation* of Q over a graph G is then defined to be the set of mappings $Q(G) = \{\mu : \mathrm{vars}(Q) \to \mathrm{dom}(G) \mid \mu(Q) \subseteq G\}$ called *solutions*, where $\mu(Q)$ denotes the image of Q under μ; that is, the result of replacing each variable $x \in \mathrm{vars}(Q)$ in Q by $\mu(x)$. Figure 4 illustrates a basic graph pattern and its evaluation over a graph, which yields three solutions. The central problem of interest to us in this paper is to compute the complete set of solutions for $Q(G)$ in optimal time and using little space.

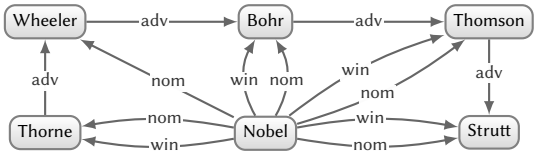
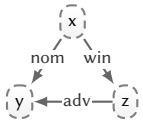


Figure 3: Graph of Nobel winners, nominees and advisors



x	y	z
Nobel	Strutt	Thomson
Nobel	Thomson	Bohr
Nobel	Wheeler	Thorne

Figure 4: Basic graph pattern (left) and its evaluation over the graph of Figure 3 (right)

2.2 Worst-case optimal joins

2.2.1 AGM bound. The AGM bound [6] defines a limit on the number of solutions for natural join queries $Q = r_1 \bowtie \ldots \bowtie r_m$, where $r_1, \ldots, r_m$ are (pairwise distinct) relation names. Given a natural join query Q and a relational instance D, the AGM bound of Q over D represents the maximum number of tuples generated by evaluating Q over any instance D' of size not greater than D.[1] If we simply assume that the size of all relations is in $O(n)$, we can speak of the AGM bound of Q, or Q^*, as a function of n.

When applying the AGM bound over graph patterns, there are three details requiring attention. First, graph patterns involve self-joins on a single ternary relation. But if we rewrite the query to make each relation name distinct, the AGM bound differs only by a constant factor. Second, graph patterns involve constants from the set $\mathcal{U}$; for example, the graph pattern Q of Figure 4 uses constants `win`, `nom` and `adv`. However, in this case we can recover the bound by transforming Q into a query Q' in which each pattern $t \in Q$ using k ($0 \leq k \leq 3$) constants is transformed into a relation of arity $3 - k$ in which we filter the base relation by the appropriate constants. Third, a triple pattern can use the same variable multiple times, but again this case can be covered by creating a relation that uses one attribute to represent the variable and that stores all tuples with the same value in the corresponding positions. Hence the same bound applies to graph patterns, within a constant factor.

2.2.2 Worst-case optimality. A *join algorithm* accepts a join query Q and a database instance D as input, and enumerates $Q(D)$ – the solutions for Q over D – as its output. A join algorithm is called *worst-case optimal* (wco) if it can run in time $O(Q^*)$. The intuition is that in the worst case a join algorithm has to enumerate Q^* results, thus taking $\Omega(Q^*)$ time. Though join algorithms do exist that run within time $O(Q^*)$ [38, 44], a logarithmic factor $O(Q^* \log n)$ is often permitted to allow more flexibility (e.g., allowing one to choose binary search over sorted relations rather than hashing [44]).

Query optimisers often convert join queries into binary join trees, where joins are evaluated pairwise using nested-loop joins, hash joins, merge joins, etc. Such approaches are not wco. Take, for example, the join query Q of Figure 1. If we first join the pair $Q_1 = \mathsf{R} \bowtie \mathsf{S}$ in order to later evaluate $Q_2 = Q_1 \bowtie \mathsf{T}$, then Q_1^* is already in the order of n^2, while Q^* is in the order of $n^{3/2}$, and hence this plan is not wco. Thus no plan involving pairwise joins can be wco. Ngo et al. [38] proposed the first wco join algorithm confirmed to run in time $O(Q^*)$. Their algorithm was later found to be outperformed by a simpler and more practical algorithm running in time $O(Q^* \log n)$, called Leapfrog TrieJoin (LTJ) [44].

2.2.3 Leapfrog TrieJoin. As illustrated in the introduction, LTJ evaluates join queries one variable-at-a-time (i.e., one attribute-at-a-time) rather than one relation-at-a-time. Here we provide more details on the LTJ algorithm in the context of graphs [23], referring to the original LTJ paper for the full relational setup [44].

Given a graph G, LTJ is built upon an abstraction for data access called a *trie-iterator*, which features one operation: `leap`.

Definition 2.1 (Trie-iterator). A *trie-iterator* for a graph G is an implementation of $\texttt{leap} : (\mathcal{U} \cup \mathcal{V})^3 \times \mathcal{V} \times \mathcal{U} \to \mathcal{U} \cup \{\bot\}$. Given a variable $x \in \mathcal{V}$, a triple pattern t with the variable x, and a constant $c \in \mathcal{U}$, $\texttt{leap}(t, x, c)$ returns the smallest constant $c_x \geq c$ from $\mathcal{U}$ such that t has solutions in G after replacing x by c_x. If there is no such value c_x, then $\texttt{leap}(t, x, c)$ returns the special value $\bot$.

Veldhuizen [44] shows that for LTJ to run in $O(Q^* \log n)$ time, it suffices for the trie iterators to support `leap` in $O(\log n)$ time (data complexity). Consider a graph pattern $Q = \{t_1, \ldots, t_m\}$, and let $v = |\text{vars}(Q)|$. For a subset $S \subseteq \text{vars}(Q)$ of variables, further let Q_S denote the set of triple patterns in Q that contain at least one variable in S. Algorithm 1 details how LTJ uses the `leap` operation to evaluate a basic graph pattern Q over a graph G. LTJ first defines an initial ordering $(x_1, \ldots, x_v)$ of vars(Q); the specific ordering does not affect wco guarantees, and will be discussed later.

Starting with x_1, LTJ finds each elimination $c \in \text{dom}(G)$ for x_1 such that, for each triple pattern $t \in Q_{\{x_1\}}$, if x_1 is replaced by c in t, then the evaluation of the modified t over G is non-empty. This is equivalent to intersecting the eliminations of x_1 for each individual triple pattern $t \in Q_{\{x_1\}}$, for which LTJ uses `leap` to recursively seek each triple pattern in $Q_{\{x_1\}}$ forward to the maximum current value for all triple patterns, returning values for which all triple patterns agree, until the eliminations for triple patterns are exhausted.

Upon finding the first elimination c of x_1, the algorithm creates a mapping $\mu = \{(x_1 \coloneqq c)\}$. Next LTJ finds values d that eliminate x_2 in $\mu(Q_{\{x_2\}})$ using the same form of intersection as before based on `leap`. When the first elimination d of x_2 is found, the current mapping is extended to $\mu = \{(x_1 \coloneqq c), (x_2 \coloneqq d)\}$. The process then continues to the next variable until all variables are eliminated, in which case μ is then a solution. If no elimination is found for a variable x_j and current mapping μ, the process then tries to eliminate x_j for the next mapping generated up to x_{j-1}. The process terminates when all mappings for x_1 have been exhausted.

We are then left to consider the implementation of `leap`. Per the name "trie-iterator", the original implementation in a relational setting was based on (virtual) tries built for each relation, with levels of the trie corresponding to attributes of a relation, and each unique root-to-leaf path encoding a tuple of the relation [44]. In a graph context, a trie would be defined with one level for subjects, one for predicates, and one for objects, and with each root-to-leaf

[1] The size of an instance D' over schema $r_1, \ldots, r_m$ is said to be not greater than an instance D if for each relation r_i, the number of tuples of r_i in D' is not greater than the number of tuples of r_i in D

Algorithm 1 LTJ for the evaluation of basic graph patterns

Input: A basic graph pattern Q, a trie-iterator $\mathcal{T}$ for a graph G, and an ordering $(x_1, \ldots, x_v)$ of the variables in vars(Q)

`leapfrog_join()`:
Output: Reports the tuples in $Q(G)$
1: **call** `leapfrog_search({}, 1)`

`leapfrog_search(`μ, j`)`:
Input: An index $1 \leq j \leq v+1$, and a mapping μ defined for the variables $\{x_k \mid k < j\}$
1: **if** $j = v+1$ **then report** μ as an output solution
2: **else**
3: $c :=$ `seek(`$\mu, j, \min \mathcal{U}$`)`
4: **while** $c \neq \perp$ **do**
5: $\mu' := \mu \cup \{(x_j := c)\}$
6: **call** `leapfrog_search(`$\mu', j+1$`)`
7: $c :=$ `seek(`$\mu, j, c+1$`)`

`seek(`μ, j, c`)`:
Input: An index $1 \leq j \leq v$, a mapping μ defined for the variables $\{x_k \mid k < j\}$, and a value $c \in \mathcal{U}$
1: Let $t_1, \ldots, t_m$ be the triple patterns in $Q_{\{x_j\}}$
2: For $1 \leq i \leq m$, let $\mu(t_i)$ be the triple pattern t_i with its variables x_k, for $k < j$, replaced by $\mu(x_k)$
3: **while** true **do**
4: **for all** $i \in [1 \,..\, m]$ **do** $c_i := \mathcal{T}$`.leap(`$\mu(t_i), x_j, c$`)`
5: **if** $\perp \in \{c_1, \ldots, c_m\}$ **then return** $\perp$
6: $c_{\mathsf{min}} := \min\{c_1, \ldots, c_m\}$; $c_{\mathsf{max}} := \max\{c_1, \ldots, c_m\}$
7: **if** $c_{\mathsf{min}} = c_{\mathsf{max}}$ **then return** c_{min}
8: $c := c_{\mathsf{max}}$

path encoding a triple of the graph. However trie-iterators based on traditional indexes (e.g., on B-trees) can only meet the $O(\log n)$-time requirement for `leap(`t, x, c`)` if the constants in t respect some predefined index order(s). In the case of graphs, supporting all possible triple patterns within the necessary time constraints implies indexing 6 different orders for all permutations of levels for subject, predicate and object (per "Flat" in Figure 2). More generally, for an arity of d, we need a total of $d!$ orders (which can be improved to $O(2^d d^{1/2})$, see Section 6): an exponential number of indexes.

In Section 3 we introduce a read-only indexing scheme for a graph G that supports `leap(`t, x, c`)` in $O(\log n)$ time, with no restrictions on the order of the constants in t, and with almost no extra space beyond that required to represent G.

2.2.4 Other wco algorithms. Recent years have seen various further proposals of wco algorithms [1, 2, 23–26, 35, 37, 40, 43]. While many such algorithms are proposed in a relational context, they can be applied over graphs represented as ternary relations. However, most such works focus on improving time, or dealing with more complex queries, rather than reducing space requirements. As an exception, Navarro et al. [35] use space close to that of the raw data by using a particular order that eliminates one bit of every variable in each round. In exchange, the time complexity includes a factor that is exponential on the arity of the relation. Some recent papers have looked at ways to combine pairwise joins with wco joins, which can help address the space requirements of the latter. Freitag et al. [19] propose a hash-based indexing scheme that can be efficiently built on-the-fly at query time, and demonstrate how to integrate these algorithms into a functioning relational database; they also propose to use wco joins only when beneficial. Our proposal avoids on-the-fly indexing in the context of graphs. Graphflow [30] integrates wco joins with pairwise joins in order to generate hybrid plans for evaluating graph queries. While their work focuses on query planning, our focus is on space-efficient indexing techniques – inspired by indexes for text – that support wco joins.

2.3 Text-indexing techniques

2.3.1 Suffix arrays. Let $T[1 \,..\, n]$ be a string over an ordered alphabet Σ, with each $T[j]$ an element in this alphabet except $T[n] = \$$, which is a special symbol larger than any other in Σ. We write $T[i \,..\, j]$ to denote $T[i] \cdots T[j]$ and $T[i \,..] = T[i \,..\, n]$. The *suffix array* A of T stores all suffixes of T in increasing lexicographic order: if one identifies each suffix $T[i \,..\, n]$ of T with the position i at which it starts, A is simply a permutation of $[n]$ that satisfies $T[A[k] \,..] < T[A[k+1] \,..]$ for all k. Intuitively, $A[k]$ indicates the index at which the k^{th} lexicographically lowest suffix of T begins. For example, if $T =$ `rococo$`, then $A = \langle 3, 5, 2, 4, 6, 1, 7 \rangle$. One use of suffix arrays is to find occurrences of a substring P in T. These occurrences, if any, correspond to an interval $A[s \,..\, e]$: they are all suffixes of T starting with P, thus forming a lexicographic range.

2.3.2 Suffix arrays for graphs. Brisaboa et al. [10] propose to use a compact representation of suffix arrays to evaluate triple patterns. For a graph G with n triples, they identify each constant in the domain of graphs with consecutive integers, and treat G as a string $T[1 \,..\, 3n+1] = s_1 p_1 o_1 \, s_2 p_2 o_2 \cdots s_n p_n o_n \, \$$. To save space, they use an index called a Compressed Suffix Array (CSA) [41]. Using space close to that of the text, this index supports traversing T (which allows one to recreate it), and searching for substrings in time comparable to what is achieved via suffix arrays. They show how to tweak the CSA of a text T representing a graph to regard the triples as cyclic, so that instead of traversing T one can jump from any third component o_i of a triple to its first component s_i. This modification allows the authors to evaluate triple patterns by finding the suffix array range $A[s \,..\, e]$ that points to the bound parts of the triple patterns, and then recovering the variable parts by simulating a traversal of the cyclic triples T pointed at within $A[s \,..\, e]$. While this work could be extended to support wco algorithms like LTJ, because their CSA-based index can efficiently support `leap` in only one direction, two orders would be needed (per Figure 2).

Our work continues the idea of using text indexes to support graph queries. However, our indexes are based on FM-indexes and wavelet trees (explained in the following), which, unlike the CSA-based index of Brisaboa et al. [10], enable the bidirectionality needed to support wco joins in just one order. Like most works using compact data structures, we assume the RAM computation model, where the typical arithmetic and logical operations on machine words of $\Theta(\log n)$ bits are carried out in constant time.

2.3.3 BWTs and FM-indexes. The *FM-index* [17] is an alternative to the CSA offering similar space and time. It is based on the *Burrows–Wheeler Transform* (*BWT*) [12] of a text T, which is a

string BWT[1 . . n] that is a permutation of T: BWT[i] = $T[A[i]-1]$, except that BWT[i] = $T[n]$ = \$ if $A[i] = 1$. Intuitively BWT[i] indicates the symbol of T directly preceding the i^{th} lexicographically lowest suffix of T (or the last symbol of T if no such symbol exists). For example, if T = rococo\$, then BWT = oorcc\$o.

Let $\Sigma \cup \{\$\} = [1, \tau]$. The BWT allows one to simulate a backwards traversal of T. Assume we know that BWT[i] corresponds to $T[j]$. Then the element corresponding to $T[j-1]$ is BWT[i'], where

$$i' = \text{LF}(i) = C[c] + \texttt{rank}_c(\text{BWT}, i), \tag{1}$$

c = BWT[i], $C[c]$ is the number of occurrences of symbols smaller than c in T, and $\texttt{rank}_c(\text{BWT}, i)$ counts the number of times c occurs in BWT[1 . . i] (we discuss its implementation soon). Thus, if we know that BWT[i] corresponds to $T[1]$, we retrieve T backwards with $T[n]$ = BWT[i], $T[n-1]$ = BWT[LF(i)], and more generally $T[n-k]$ = BWT[$\text{LF}^k(i)$]. The backward traversal of T is thus achieved by repeated applications of the function LF, called LF-steps. For example, with T = rococo\$ and BWT = oorcc\$o, if we know that BWT[2] refers to $T[4]$ = o, then BWT[LF(2)] = BWT[2 + 2] = BWT[4] corresponds to the element $T[3]$ = c.

The FM-index uses the BWT to simulate searches in the suffix array. Given a string $P[1 . . m]$, the goal of this search is to find the maximal interval $A[s_P . . e_P]$ such that the suffixes of the text represented by this interval are all prefixed by P. This is done via a process called *backward search* as follows. The interval $A[s_{P[m]} . . e_{P[m]}]$ of $P[m]$ is $s_{P[m]} := C[P[m]] + 1$ and $e_{P[m]} := C[P[m]+1]$. Assume that we know the interval corresponding to the occurrences of $P[i+1 . . m]$. Then, we can find the interval for $P[i . . m]$ as follows:

$$\begin{aligned} s_{P[i..m]} &:= C[P[i]] + \texttt{rank}_{P[i]}(\text{BWT}, s_{P[i+1..m]} - 1) + 1, \\ e_{P[i..m]} &:= C[P[i]] + \texttt{rank}_{P[i]}(\text{BWT}, e_{P[i+1..m]}). \end{aligned} \tag{2}$$

This simulates taking simultaneous LF-steps at all the positions $k \in [s_{P[i+1..m]} . . e_{P[i+1..m]}]$ where BWT[k] = $P[i]$: the suffixes starting with $P[i . . m]$ are obtained from those starting with $P[i+1 . . m]$ and preceded by $P[i]$. If for all $i \in [1 . . m]$ we have $s_{P[i..m]} \leq e_{P[i..m]}$, then the range for P is the final $A[s_P . . e_P]$. For example, given T = rococo\$, BWT = oorcc\$o, P = oco, and $A = \langle 3, 5, 2, 4, 6, 1, 7\rangle$, we can find the interval for $P[1 . . 3]$ as follows:

$$\begin{aligned} s_{P[3]} &:= C[\texttt{o}] + 1 &&= 3 \\ e_{P[3]} &:= C[\texttt{p}] &&= 5 \\ s_{P[2..3]} &:= C[\texttt{c}] + \texttt{rank}_{\texttt{c}}(\text{BWT}, s_{P[3]} - 1) + 1 &&= 1 \\ e_{P[2..3]} &:= C[\texttt{c}] + \texttt{rank}_{\texttt{c}}(\text{BWT}, e_{P[3]}) &&= 2 \\ s_{P[1..3]} &:= C[\texttt{o}] + \texttt{rank}_{\texttt{o}}(\text{BWT}, s_{P[2..3]} - 1) + 1 &&= 3 \\ e_{P[1..3]} &:= C[\texttt{o}] + \texttt{rank}_{\texttt{o}}(\text{BWT}, e_{P[2..3]}) &&= 4 \end{aligned}$$

Indeed, the occurrences of P in T start at $T[A[3] . . 7] = T[2 . . 7]$ = ococo\$ and $T[A[4] . . 7] = T[4 . . 7]$ = oco\$.

The space required by an FM-index is that used for C and BWT. We can always store C within $n + \tau + o(n+\tau)$ bits, as a bitvector $D[1 . . n+\tau]$ with 1's at positions $i + C[i]$; then $C[i]$ is computed as $\texttt{select}_1(D, i) - i$. This operation finds the position of the i^{th} 1 in D, and is computed in constant time by storing $o(|D|)$ bits on top of D [13, 31]. Regarding BWT, most implementations use a *wavelet tree*, which requires space close to that of a plain representation of BWT, and can solve $\texttt{rank}_c(\text{BWT}, i)$ queries in time logarithmic on the alphabet size τ [18], enabling search of P in $O(m \log \tau)$ time.

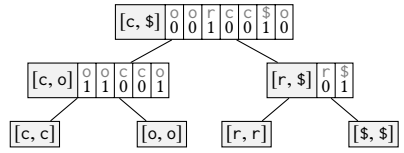


Figure 5: Example wavelet tree for oorcc\$o

2.3.4 Wavelet trees. A wavelet tree [22, 34] is a binary tree that represents a string $S[1 . . n]$ from an alphabet $[1, \tau]$ using $n \log_2 \tau + o(n \log \tau)$ bits of space (and can use compressed space). Each node represents a range of alphabet symbols. The root represents $[1, \tau]$ and the c^{th} left-to-right leaf represents $[c, c]$, i.e., the interval containing only the c^{th} character, which we denote as c. If an internal node represents $[a, b]$, then its left child represents $[a, \lfloor (a+b)/2 \rfloor]$ and its right child represents $[\lfloor (a+b)/2 \rfloor + 1, b]$. Conceptually, a node representing $[a, b]$ stores the subsequence $S_{a,b}[1 . . n_{a,b}]$ of S formed by the symbols in $[a, b]$. In practice, this node only stores a bitvector $B_{a,b}[1 . . n_{a,b}]$ with a 0 at position i if $S_{a,b}[i] \leq (a+b)/2$ and a 1 otherwise (marking whether $S_{a,b}[i]$ belongs to its left or right child). The leaves store nothing. Figure 5 exemplifies the wavelet tree for oorcc\$o; only the bitvectors and pointers are stored, where other elements are included for illustration purposes.

Regarding space, in a wavelet tree for a string $S[1 . . n]$, the total number of bits stored at the nodes of each non-leaf level is also n. Since the tree has $\lceil \log_2 \tau \rceil$ non-leaf levels, storing all the bitvectors requires at most $n\lceil \log_2 \tau \rceil$ bits, just like a plain representation of S. To support operations efficiently (e.g., accessing $S[i]$), all the bitvectors $B_{a,b}$ are enhanced with $o(n_{a,b})$ further bits so that operations $\texttt{rank}_b(B_{a,b}, i)$ and $\texttt{select}_b(B_{a,b}, i)$ can be answered in constant time [13, 31]. This adds $o(n \log \tau)$ further bits of space. The pointers of the tree add $O(\tau \log n)$ bits to its space usage (not negligible for large τ). This is avoided in a pointerless variant called a *wavelet matrix* [14], which offers the same functionality in less space.

Operations access, rank, and select are solved by traversing one root-to-leaf path in the wavelet tree, in $O(\log \tau)$ time. As an example, let us show how to access $S[i]$: we start with $[a, b] := [1, \tau]$ and $i' := i$ at the root. If $B_{a,b}[i'] = 0$, we set $b := \lfloor (a+b)/2 \rfloor$, $i' := \texttt{rank}_0(B_{a,b}, i')$, and continue by the left child. Otherwise, we set $a := \lfloor (a+b)/2 \rfloor + 1$, $i' := \texttt{rank}_1(B_{a,b}, i')$, and continue by the right child. When we arrive at a leaf, it holds $a = b = S[i]$, and moreover $\texttt{rank}_{S[i]}(S, i) = i'$ (this is used in the LF formula, Eq. (1)).

For example, we can compute BWT[7] (whose wavelet tree is shown in Figure 5) by reading $B_{\texttt{c},\$}[i' = 7] = 0$ in the root bitvector $B_{\texttt{c},\$} = 0010010$, thus going left with $i' := \texttt{rank}_0(B_{\texttt{c},\$}, 7) = 5$. On the left child of the root, we read $B_{\texttt{c},\texttt{o}}[i' = 5] = 1$, so we go right with $i' := \texttt{rank}_1(B_{\texttt{c},\texttt{o}}, 5) = 3$. We then arrive at the leaf [o, o], so we know that BWT[7] = o and $\texttt{rank}_{\texttt{o}}(\text{BWT}, 7) = i' = 3$.

As a more advanced operation, the wavelet tree can return all of the distinct values c in a range $S[s . . e]$, also computing $[s_c, e_c] = [\texttt{rank}_c(S, s-1) + 1, \texttt{rank}_c(S, e)]$ for each. The algorithm takes time $O(k \log(\tau/k))$ to report the k distinct values in $S[s . . e]$ [21].

Finally, wavelet trees support the *range-next-value* operation in time $O(\log \tau)$ [21], which we can use for list intersection. Specifically, given a range $S[s . . e]$ and a threshold $c_x \in [1, \tau]$, we wish to

find the smallest symbol $c \geq c_x$ that occurs in $S[s\,..\,e]$. The algorithm also finds the range $[s_c\,..\,e_c]$ so that $S[s\,..\,e]$ contains from the s_c^{th} to the e_c^{th} occurrence of c in S (i.e., $s_c = \mathsf{rank}_c(S, s-1)+1$ and $s_e = \mathsf{rank}_c(S, e)$, similar to the backward search in Eq. (2)).

3 One ring to index them all

We now present our ring index scheme for a graph G, and describe how to implement the trie-iterator interface over it. This allows LTJ to evaluate basic graph patterns over G in wco time using almost no space beyond that of a raw, and even a compressed, representation.

3.1 The BWT of cyclic triples

We map all the constants in $\text{dom}(G)$ to consecutive integers $[1\,..\,U]$. We then build our index on the resulting set of n integer triples (s, p, o). We add U to all the values p and $2U$ to all the values o, so that all the second components are larger than all the first ones, and all the third components are larger than all the second ones. Finally, we sort the resulting triples lexicographically (i.e., by subject, ties broken with predicates, and then ties broken with objects) to obtain the ordered triples (s_i, p_i, o_i). We then define the *text*

$$T[1\,..\,3n+1] \;=\; s_1 p_1 o_1\, s_2 p_2 o_2 \cdots s_n p_n o_n\, \$$$

concatenating each triple and then \$ (the largest symbol of all).

3.1.1 Bended BWT. Next, we build the BWT of T and slightly modify it so that when we try to find the symbol that precedes s_i, it returns o_i instead of o_{i-1} (or instead of \$ if $i = 1$), thus regarding triples as cyclic. Loosely speaking, we do this by shifting the objects in BWT so that o_i is now in o_{i-1}'s place. In the following we write $X \cdot Y$ to denote the concatenation of two strings X and Y.

Definition 3.1. The *bended BWT* of $\text{BWT}[1\,..\,3n+1]$ is

$$\text{BWT}^*[1\,..\,3n] = \text{BWT}[2\,..\,n] \cdot \text{BWT}[3n+1] \cdot \text{BWT}[n+1\,..\,3n].$$

Let A be the suffix array of T. The following properties of A and BWT are useful to understand the bended BWT^*:

(1) Because $s_i < p_j < o_k$ for all i, j, k, all the suffixes starting with subjects (i.e., of the form $T[3r+1\,..]$ for $0 \leq r < n$) precede in A all those starting with predicates (i.e., $T[3r+2\,..]$), and those precede all the suffixes starting with objects (i.e., $T[3r+3\,..]$). The lexicographically largest suffix is $T[3n+1\,..] = \$$, so $A[3n+1] = 3n+1$ and thus $\text{BWT}[3n+1] = o_n$.
(2) Because the triples are sorted in T, every suffix starting with s_i is lexicographically smaller than the one starting with s_{i+1}, and thus it holds that $A[i] = 3(i-1)+1$ for all $1 \leq i \leq n$. Therefore, $\text{BWT}[i] = o_{i-1}$ for $2 \leq i \leq n$ and $\text{BWT}[1] = \$$.

These two properties, and the way in which BWT^* is obtained from BWT, imply that BWT^* is divided into three zones: the objects, positioned in $[1\,..\,n]$, the subjects, appearing in $[n+1\,..\,2n]$, and then the predicates in positions $[2n+1\,..\,3n]$. Moreover, since $\text{BWT}[2\,..\,n] = o_1 \cdots o_{n-1}$ and $\text{BWT}[3n+1] = o_n$, the zone of objects is exactly $o_1 \cdots o_n$; that is, $\text{BWT}^*[n]$ denotes the object of the n^{th} triple in the order. This means that BWT^* is of the form

$$\text{BWT}^* \;=\; (o_1 \cdots o_n) \cdot (\text{subjects}) \cdot (\text{predicates}). \tag{3}$$

Example 3.2. Figure 6 shows the index corresponding to the graph of Figure 3. On the top-left, we map $\text{dom}(G)$ to the interval $[1\,..\,9]$. The top-center shows the resulting set of $n = 13$ triples (e.g. (Bohr,adv,Thompson) becomes $(1, 7, 3)$). On the top-right we see the result of shifting the predicates by $U = 9$ and the objects by $2U = 18$. After sorting the shifted triples, we concatenate them and append \$ to form the text T, where we distinguish identifiers of subjects in blue, of predicates in red, and of objects in green.

Below T we show its suffix array A, which is largely divided into zones pointing to subjects, to predicates, and to objects (and $A[40]$ points to the suffix \$). We show how C, below A, identifies the areas of the suffixes starting with each different identifier. For example, the suffixes $A[C[6]+1\,..\,C[6+1]] = A[5\,..\,13] = \langle 13, 16, 19, 22, 25, 28, 31, 34, 37\rangle$ are those starting with 6.

Below A, we show BWT, which for each q gives $T[A[q]-1]$. Again, disregarding its extremes, it is divided into three zones, composed only of object, subject, and predicate identifiers, left to right. Below it, we show BWT^*, the bended BWT.

The figure shows how we recover the first triple, $(1, 16, 21)$ (the shifted version of $(1, 7, 3)$) by starting from position 1. Since $\text{BWT}^*[1] = 21$, we know that the object is 21. It is the first 21 (i.e., $\mathsf{rank}_{21}(\text{BWT}^*, 1) = 1$), so we go to the next position after $C[21] = 32$ (i.e., $\text{LF}^*(1) = 33$), to find the corresponding predicate at $\text{BWT}^*[33] = 16$. Since $\mathsf{rank}_{16}(\text{BWT}^*, 33) = 3$, we compute $\text{LF}^*(33) = C[16]+3 = 16$ and find the subject at $\text{BWT}^*[16] = 1$. Note also that $\text{LF}^*(16) = 1$ because $C[1] = 0$ and $\mathsf{rank}_1(\text{BWT}^*, 16) = 1$, so if we keep going we cycle over the triple. To find the n^{th} triple, we apply the same process starting with $\text{BWT}^*[n]$. □

The next lemma, proved in the supplementary material, shows that this cyclical traversal always occurs, i.e., that our bended BWT is precisely the *Extended BWT* [29] of our sorted triples.

Lemma 3.3. BWT^* *sees* $T[1\,..\,3n]$ *as the concatenation of* n *cyclic triples: calling* LF^* *its LF function according to Eq. (1), for every* $1 \leq t \leq n$*, where* $T[A[t]\,..] = s_t p_t o_t \cdots$*, it holds that* $T[A[\text{LF}^*(t)]] = o_t$*,* $T[A[\text{LF}^*(\text{LF}^*(t))]] = p_t$*, and* $\text{LF}^*(\text{LF}^*(\text{LF}^*(t))) = t$.

Backward search then works on the set of cyclic triples as well, because each step simulates a batch of LF^* operations. In the sequel we use BWT to denote its bended variant, BWT^*, for simplicity.

3.1.2 Space requirements. The ring index of the n triples is composed of the wavelet tree of the bended BWT plus its C array to support LF-steps and backward searches. We note that the BWT representation is not *in addition* to the raw data, but it *replaces* the raw data, because we can retrieve the i^{th} triple as

$$(\text{BWT}[\text{LF}(\text{LF}(i))], \text{BWT}[\text{LF}(i)] - U, \text{BWT}[i] - 2U),$$

where the wavelet trees compute $\text{LF}(\cdot)$ and $\text{BWT}[\cdot]$ in time $O(\log U)$.

The representation of the bended BWT using wavelet trees requires $3n\log_2(3U) + o(n\log U)$ bits; note $3n\log_2(3U) = 3n\log_2 U + O(n)$. Another $O(n)$ bits are required for the bitvector D. Note that $O(n) \subset o(n\log U)$ because the n triples are distinct and thus $U = \Omega(n^{1/3})$. The total space used by the ring index is then $3n\log_2 U + o(n\log U)$ bits, which is only sublinear space on top of the $3n\log_2 U$ bits required for the raw representation of the triples.

The BWT can be computed (and bended) in $O(n)$ time and $O(n\log U)$ bits of working space [8]. Its wavelet tree is then built in time $O(n\log U/\sqrt{\log n})$ [33], which is $O(n\sqrt{\log U})$ as $U \leq n$.

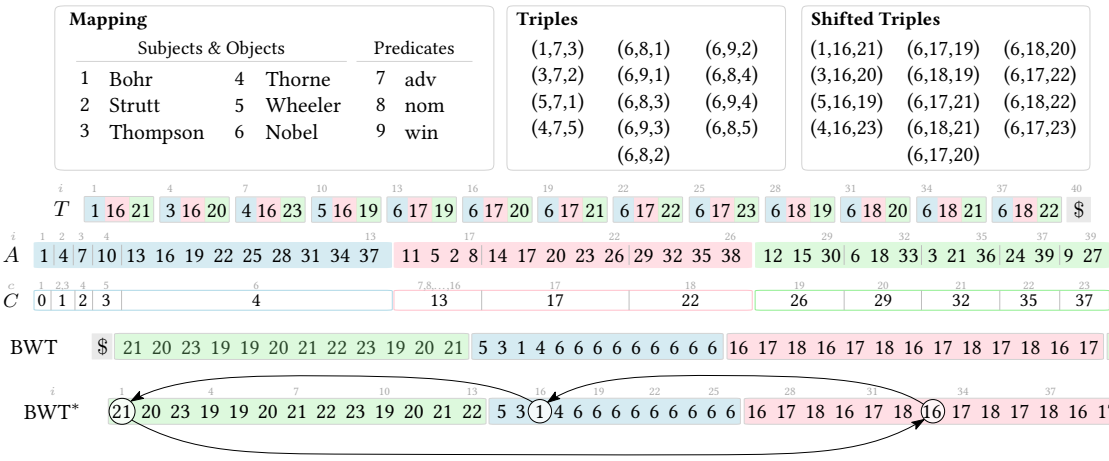


Figure 6: Our ring index for the graph of Figure 3

Theorem 3.4. *Let $|G| = 3n \log_2 U$ be the space in bits of a raw representation of a graph G with n edges and $U = |\mathrm{dom}(G)|$ different identifiers. Then the ring index of G uses $|G| + o(|G|)$ bits of space, and can retrieve any desired edge in $O(\log U)$ time. The ring index is built in $O(n\sqrt{\log U})$ time within $O(|G|)$ bits of working space.*

This space is worst-case. Sections 4 and 5 show that we can make the ring index use space close to a *compressed* representation of T (we explain why in the supplementary material).

3.2 Processing joins

We now describe how to support `leap` over the BWT representation of a graph in $O(\log U)$ time, building on the algorithms described in Section 2.3.4. This running time for `leap` implies the worst-case optimality of LTJ (Algorithm 1) over our representation [44]. Thus, in this section we prove the following theorem.

Theorem 3.5. *On a graph G, our ring index evaluates a basic graph pattern Q formed by m triple patterns in $O(Q^* \cdot m \log |\mathrm{dom}(G)|)$ time, where Q^* is the maximum possible output (AGM bound) of such a query on some graph of size $|G|$. The working space of the query algorithm is $O(1 + |\mathrm{var}(Q)|)$ words.*

Let $Q = \{t_1, \ldots, t_m\}$ be a basic graph pattern. In this section we assume that the constants in each t_i have been appropriately transformed into integers and shifted, summing U to the predicate constants, and $2U$ to the object constants in each t_i, as is done for the BWT representation of G (see Figure 6). We further assume that no variable appears more than once in a triple pattern t_i; the other case is discussed in the supplementary material. We use $t(G)$ as shorthand for the evaluation of the (singleton) basic graph pattern $\{t\}(G)$ (see Section 2.1). We denote by $G_t = \{\mu(t) \mid \mu \in t(G)\}$ the *occurrences* of t in G, that is, the set of triples in G matching t.

3.2.1 Computing the occurrences of a triple pattern. Because the constants in a triple pattern t are always consecutive when t is regarded as cyclic, there is a range $\mathrm{BWT}[s \,..\, e]$ such that $A[s \,..\, e]$ points to all the subjects, all the predicates, or all the objects of G_t. The following lemma, proved in the supplementary material, shows how to find $[s \,..\, e]$ in $O(\log U)$ time.

Lemma 3.6. *Let $t = (\alpha, \beta, \gamma)$ be a triple pattern with $0 \le b \le 3$ constants. In $O(\log U)$ time, we can find values s, e such that:*

- *If $|G_t| = 0$, then $s = e = \perp$; otherwise*
- *$e = s + |G_t| - 1$, and $A[s \,..\, e]$ points in T to either the subjects, the predicates, or the objects in G_t, whichever is the first following a variable in t if $b < 3$; in particular,*
  - *if $b = 1$, and d is the constant in t, then $T[A[k]] = d$ $\forall k \in [s \,..\, e]$.*

3.2.2 Supporting leaps. We show how to support $\texttt{leap}(t'_i, x, c)$, where t'_i is either a triple pattern t_i from Q, or one of its progressively bound versions $\mu(t_i)$ in Algorithm 1.

To evaluate $\texttt{leap}(t'_i, x, c)$, we start by obtaining the values s_i, e_i of Lemma 3.6 for t_i. If $s_i, e_i = \perp$ we simply return $\perp$. Otherwise, let $t'_i = (\alpha_i, \beta_i, \gamma_i)$. We search for $c_x \ge c$ (recall Definition 2.1) differently depending on where x and the constants in t_i are. If only α_i is a constant and $\beta_i = x$, or only β_i is a constant and $\gamma_i = x$, or only γ_i is a constant and $\alpha_i = x$, then we will find c_x *forwards*, because x follows the part of the triple pattern that is already bound. Otherwise, we will find c_x *backwards*, that is, looking for x preceding the substring that is already bound.

When we process a triple pattern $t'_i = (\alpha_i, \beta_i, \gamma_i)$ backwards, we have the range $A[s_i \,..\, e_i]$ pointing to the first constant of the occurrences of t'_i, and want to find the smallest $c_x \ge c$ such that some of the (cyclic) suffixes $T[A[k] \,..]$, $s_i \le k \le e_i$, are preceded by c_x. Equivalently, we want to find the smallest $c_x \ge c$ in $\mathrm{BWT}[s_i \,..\, e_i]$. This corresponds to the range-next-value operation of Section 2.3.4, which is supported in $O(\log U)$ time by the wavelet tree of BWT.

When we rather process $(\alpha_i, \beta_i, \gamma_i)$ forwards, only one of α_i, β_i, or γ_i, is bound. By Lemma 3.6, we know that $T[A[k]] = d$ for all $s_i \le k \le e_i$. We must thus find the smallest $c_x \ge c$ that follows some of those ds in T, i.e., $T[A[k] + 1] = c_x$. We solve this by, first, finding the leftmost occurrence of d in $\mathrm{BWT}[C[c] + 1 \,..]$, where the suffixes in A start with a symbol c or larger. Such an occurrence q is found in $O(\log U)$ time using the wavelet tree, by making $p := \texttt{rank}_d(\mathrm{BWT}, C[c])$ and $q := \texttt{select}_d(\mathrm{BWT}, p + 1)$. This is the position in A of the leftmost suffix $A[q]$ that starts with

a symbol $c_x \geq c$ and is preceded by d. To find the value of c_x, we binary search for q in C, looking for $C[c_x] < q \leq C[c_x + 1]$.[2]

Lemma 3.7. *Let G be a graph, t be a triple pattern, x be a variable that appears exactly once in t, and $c \in U$ a constant. Then the* BWT *representation of G supports* `leap(`t, x, c`)` *in $O(\log U)$ time.*

Finally, note that Algorithm 1 can be modified so that, except for μ, the working space used by `seek` is constant (i.e., only maintaining $c_{\min}$ and $c_{\max}$, not every c_i); the same holds for `leap` and leapfrog_search. We can further maintain μ in constant space per recursive call by storing only the last assignment and pointing to the previous one in the stack. Thus, the working space of Algorithm 1 is $O(v + 1)$, since the maximum stack height is $v = |\text{vars}(Q)|$.

4 Engineering and implementation

We describe in this section the most relevant engineering and implementation aspects of our ring index. Our implementation is available for reviewing online [5].

4.1 Representation

In order to reduce the size of the universe, we work with a representation that is slightly more involved. Following Eq. (3), we cut the BWT into three components:

$$\text{BWT}[1\,..\,3n] \;=\; \text{BWT}_o[1\,..\,n] \cdot \text{BWT}_s[1\,..\,n] \cdot \text{BWT}_p[1\,..\,n],$$

where BWT_o is formed only by objects, BWT_s only by subjects, and BWT_p only by predicates. With a little more work and three independent C arrays, C_o, C_s, and C_p, we can use the identifiers in non-shifted form (i.e., without adding U and $2U$). This reduces the alphabet size, which improves both space and operation time for the wavelet trees. The mapping from $\text{dom}(G)$ to consecutive integers is done by sorting the triples by predicate (using `std::sort`), and then hashing subjects and objects to ensure uniqueness.[3]

4.2 Algorithm

We implement Algorithm 1 with some improvements on the description of Section 3.2. First, for each t'_i we maintain the values s_i, e_i instead of computing them from scratch during each `leap`. We find s_i, e_i at the beginning of `leapfrog_search` as described in Lemma 3.6, and then update them after each further binding of t'_i. The working space then rises to $O(mv)$, which is still very low.

The second optimisation is to handle the *lonely variables* [23], i.e., variables that appear in only one triple pattern t_i, in a different way: once the other variables of t_i have been bound, we report all the possible bindings of our ranges. From the current values s_i, e_i, we bind the remaining variables backwards, one by one. For each variable x, this corresponds to finding all the distinct values in $\text{BWT}[s_i\,..\,e_i]$, which is done as described in Section 2.3.4.

4.3 Variable elimination order

The running time of LTJ can often sharply depend on selecting a good order in which variables are eliminated. It turns out that our ring index can also be used to provide relevant statistics on the fly, computed in logarithmic time, and without additional profiling.

Recall that given a triple pattern t_i, our index quickly computes the initial ranges $A[s_i\,..\,e_i]$, so that the number of triples matching the pattern is exactly $e_i - s_i + 1$. We use this to construct the following elimination order for variables that appear in more than one triple pattern. First we compute $c(t_i) = (e_i - s_i + 1)/n$ for each such triple t_i. Then we estimate the cardinality of each variable x as $c_{\min}(x) = \min_{t \in Q_{\{x\}}} c(t)$. The variables x are then bound by increasing order of $c_{\min}(x)$, but ensuring that each new variable shares a triple pattern with some previous one, if possible.

4.4 Indexing

We implemented the ring index in C++11 over the succinct data structures library, sdsl (https://github.com/simongog/sdsl-lite). To construct the indexes, we first build the suffix array A with quicksort, and then extract the BWTs. Because the alphabets are generally large, we implemented the wavelet trees as wavelet matrices [14]. We provide two flavours of indexes, with the bitvectors of the wavelet matrices stored either in plain or compressed form. The compressed bitvectors enable high-order entropy compression of the BWT [27]. These bitvectors use a parameter b in sdsl; larger values for b offer better compression but slower operations.

5 Experimental results

We now compare our system – running a modified version of LTJ over a ring index – versus state-of-the-art alternatives in terms of the space used for indexing and the time for evaluating basic graph patterns. We expect that our system will use less space than non-compact alternatives, and that it will use less time for evaluating queries than non-wco alternatives, while remaining competitive with wco alternatives. We further compare compressed and uncompressed variants of the ring index, where we expect the compressed variant to use less space but to have slower query times.

We run two benchmarks over the Wikidata graph [45], which we choose for its scale, diversity, prominence, data model (it has labelled edges) and real-world query logs [9, 28]. The first benchmark is the Wikidata Graph Pattern Benchmark (WGPB) proposed by Hogan et al. [23] for a sub-graph of Wikidata, with diverse graph patterns. The second benchmark evaluates real-world graph patterns extracted from Wikidata query logs at full scale.

We begin by describing our setup; further details for reproducing experiments are given in the online material [5].

5.1 Experimental setup

Our experiments compare various wco algorithms, as follows.

Ring and C-Ring: LTJ running over our ring index using plain and compressed bitvectors, respectively. The latter uses parameter $b = 16$. The system operates in main memory.

EmptyHeaded: An implementation [1] of NPPR [38], which is an instance of the same wco algorithm as LTJ [39]. Triples are stored as 6 different tries (all orders) in main memory.

Graphflow: A graph query engine that indexes property graphs using in-memory sorted adjacency lists and supports hybrid plans blending wco and pairwise joins [30].

[2] As mentioned in Section 2.3.3, C might be stored as a bitvector to save space for large alphabets. In this case the binary search is replaced by $c_x = \texttt{select}_0(D, q) - q$.

[3] In initial experiments with different alphabet orders, we found minor space benefits only for the compressed version of the ring. An interesting future direction would be to explore optimisations based on carefully choosing this order [4].

Qdag: The only previous succinct wco index [35], based on a quadtree representation of the graph that runs in main memory.

We disregard other compressed graph indexes [3, 10] that support only single triple patterns or pairwise joins. For reference we further include results for prominent graph database systems:

Jena: A reference implementation of the SPARQL standard. We use TDB, with B+-trees indexes in three orders: *spo*, *pos*, and *osp*. The system supports nested-loop joins.

Jena LTJ: An implementation [23] of LTJ on top of Jena TDB. All six different orders on triples are indexed in B+-trees.

RDF-3X: The reference scheme [36] that indexes a single table of triples in a compressed clustered B+-tree. The triples are sorted, so that those in each B+-tree leaf can be differentially encoded. RDF-3X handles triple patterns by scanning ranges of triples and uses a query optimiser based on pairwise joins.

Virtuoso: A widely used graph database hosting the public DBpedia endpoint, among others [15]. It provides a column-wise index of quads with an additional graph (g) attribute, with two full orders (*psog*, *posg*) and three partial indexes (*so*, *op*, *gs*) optimised for patterns with constant predicates. The system supports nested loop joins and hash joins.

Blazegraph: The graph database system [42], hosting the official Wikidata Query Service [28]. We run the system in triples mode wherein B+-trees index three orders: *spo*, *pos*, and *osp*. The system supports nested-loop joins and hash joins.

The wco algorithms – Ring, C-Ring, EmptyHeaded, Graphflow and Qdag – work in memory with dictionary-encoded numeric constants in the queries, graphs, and results, while the database systems – Blazegraph, Jena, Jena-LTJ, RDF-3X and Virtuoso – may use secondary storage and work with strings as constants in the queries, graphs, and results (though internally they may use dictionary-encoded numeric constants). The results for these popular database systems are included as a point of reference; we discuss the overhead of dictionary decoding/encoding in the supplementary material.

We run our experiments on an Intel(R) Xeon(R) CPU E5-2630 at 2.30GHz, with 6 cores, 15 MB of cache, and 96 GB of RAM. Our code was compiled using g++ with flags `-std=c++11`, `-O3`, and `-msse4.2`. Systems are then configured per vendor recommendations [5]. All queries are run with a timeout of 10 minutes and a limit of 1000 results (as originally proposed for WGPB [23]).

5.2 Graph patterns benchmark

We first run the Wikidata Graph Pattern Benchmark (WGPB) [23], which uses a Wikidata sub-graph with $n = 81{,}426{,}573$ triples, 19,227,372 subjects, 2,101 predicates, and 37,641,486 objects. The benchmark provides 17 query patterns of different widths and shapes, including cyclic and acyclic queries, as shown in Figure 7. Each pattern is instantiated with 50 queries built using random walks such that the results are nonempty. The benchmark allows us to compare different alternatives for different abstract patterns. All predicates are constant, all subjects and objects are variables, and each variable appears at most once in the same triple pattern.

5.2.1 Indexing and space. There are 4,869,562 identifiers that are both subjects and objects in the graph, so we use a common alphabet for both of size $|S| = |O| = 51{,}999{,}296$. Our ring index, in either variant, was built in 12.6 minutes, that is, at a rate of around 6.4 million triples per minute. The working space used for indexing was 2.3 GB, about twice the size of the text T of triples (which uses 932 MB). The construction of the BWTs takes about a minute; the rest is spent in building the wavelet matrices.

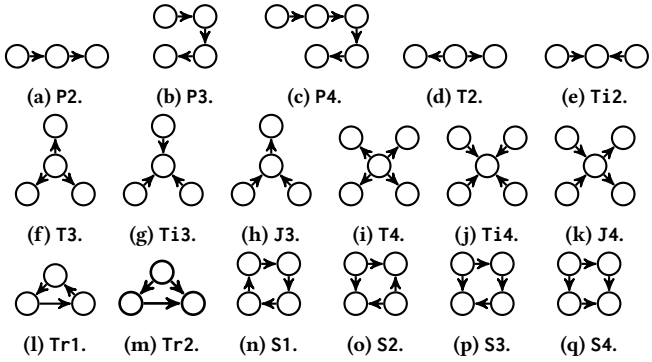


Figure 7: Query patterns for the Wikidata benchmark

Let us contextualise the space used by the ring index. A simple representation of the dataset using 32-bit integers for all the values requires 12 bytes per triple. A *packed* representation requires $\log_2\lceil|S|\rceil + \log_2\lceil|P|\rceil + \log_2\lceil|O|\rceil = 26 + 12 + 26 = 64$ bits, or 8 bytes, per triple. Our ring index with plain bitvectors privileges time performance, and thus its `rank` and `select` structures pose a 57% space overhead, which adding the bitvectors D sums up to 12.70 bytes per triple, very close to the size of the simple representation. Instead, our ring index using compressed bitvectors with $b = 16$ requires 6.68 bytes per triple, less than the packed representation.

As explained, our index replaces the representation of the triples, because one can obtain any desired triple from the index. The time to retrieve any arbitrary triple is 5 microseconds with plain bitvectors and 20 microseconds with compressed bitvectors.

Table 1 compares the space of the indexes, showing that even the uncompressed Ring uses 5.7–142 times less space than all other non-ring indexes except Qdag, which is also succinct. Qdag, however, uses 32% more space than C-Ring.

Graphflow failed to index the graph both on the experimental machine (with 96 GB of RAM), and another machine on which it was assigned 680 GiB (~730 GB) of Java heap space. Reviewing the source code, Graphflow loads in-memory adjacency lists with $p \times v$ arrays of 32-bit integers, where p is the number of unique predicates (i.e., edge labels) and v is the number of unique subjects/objects (i.e., nodes), thus requiring $\Omega(pv)$ space. Hence it was not feasible to load the Wikidata graph for which $p = 2{,}101$ and $v = 51{,}999{,}296$. The system rather targets property graphs with few edge labels (unique predicates) and does not support queries with node identifiers (constant subjects and objects, as needed for our second set of experiments, described presently).[4] We conclude that Ring and C-Ring occupy much less space than Graphflow, but comparison of query runtimes was not possible for the selected experiments due to insufficient RAM.

One may compare the space of C-Ring with plain data compressors applied to the text T of triples. We thus built an additional

[4] From personal communication with the first author of the Graphflow paper [30].

Table 1: Index space, in bytes per triple, on the Wikidata subgraph, and average query time on WGPB data and queries

System (Data + Indexes)	Space	Time (msec)
Ring	12.70	31
C-Ring	6.68	97
EmptyHeaded	1,809.84	118
Graphflow	>8,966.90	—
Qdag	8.86	14,873
Jena	72.32	127
Jena LTJ	144.64	59
RDF-3X	107.65	182
Virtuoso	104.49	1,135
Blazegraph	99.86	1,709

C-Ring variant with $b = 64$, which uses just 5.35 bytes per triple, close to the space obtained by typical compressors like `gzip -9` (4.83), `bzip2 -9` (4.78) and `ppmdj` (4.61). Stronger compressors obtain considerably less space, such as `rar -m5` (3.52), `p7zip` (2.67), and a special-purpose front-coding plus δ-coding of the differences that we implemented inspired in the RDF-3X encoding (1.87). We do not run queries on C-Ring with $b = 64$ because it is considerably slower, for example it takes 73 microseconds to retrieve an arbitrary triple. Still, it can be regarded as a reasonable compression format that supports random access and even (slow) queries.

5.2.2 Query times. Table 1 gives the average time taken by each index to sequentially evaluate all the queries. Ring is the fastest index overall, outperforming EmptyHeaded (which uses 142 times more space) by a slight margin[5] and all the other non-succinct indexes by a factor of 2–36. C-Ring is about three times slower than Ring, but it is the smallest index, using about half the space of Ring while still offering faster runtimes than most other indexes. In particular, it is 150 times faster than Qdag, the next smallest index.[6]

Figure 8 shows the distribution of the times for the 17 query types. For readability, the plots omit Jena, which was always slower than Jena-LTJ, showing the benefits for query times of wco joins (per Table 1, Jena-LTJ is about twice as fast overall versus Jena). Considering the median values, Ring is the fastest in 8 of the 17 patterns, Qdag in 5, and EmptyHeaded in the other 4.

Ring is the best, or near the best, in all acyclic queries except for the short paths of `P2`. Qdag is the best, or near the best, in all queries with just three variables (`P2`, `T2`, `Ti2`, `Tr1`, `Tr2`), but has problems with some of the larger queries, particularly acyclic ones. EmptyHeaded is the best, or near the best, in all cyclic queries, and in some acyclic queries, but has bad cases for long paths (`P4`) and some tree-like queries (`T3`, `T4`, `J4`)). More generally, we observe that Ring offers more stable query times than EmptyHeaded and Qdag: the 75% percentile never exceeds 0.05 seconds.

[5]EmptyHeaded does not support limiting the number of results to 1,000. When computing unlimited results, EmptyHeaded outperforms all the other indexes except Ring, which is still faster: 110 vs. 118 milliseconds for EmptyHeaded (Table 1). The absence of this limit is less noticeable in Figure 8, as very few queries exceed 1,000 results.

[6]Qdag does not handle constants in the triple patterns. Since these queries have constant predicates, we use a Qdag to index one binary relation per predicate.

Table 2: Index space (in bytes per triple) and some statistics on the query times (in seconds) on the full Wikidata graph. The last column counts the queries taking over 10 minutes

System	Space	Min	Avg	Median	Timeouts
Ring	13.86	10^{-5}	3.920	0.021	5
Jena	95.83	0.002	11.513	0.035	19
Jena LTJ	168.84	0.003	1.939	0.162	1
RDF-3X	85.73	0.007	8.239	0.126	13
Virtuoso	60.07	0.002	4.882	0.050	8
Blazegraph	90.79	0.008	9.220	0.054	14

Comparing Ring with Qdag, the latter uses an encoding that grows exponentially with the number of nodes in patterns [35], so the gains of Ring for larger queries is expected. Comparing Ring with EmptyHeaded, we speculate that the advantage of Ring in acyclic queries is mostly due to the lonely variables optimisation. To be more precise, consider queries in `T4`, `Ti4`, `J4`. In the case of Ring, once the central variable connecting the other four variables is eliminated, the remaining variables are subsequently eliminated using the lonely variables optimisation (see Section 4.2). EmptyHeaded rather processes acyclic queries using a version of the traditional Yannakakis algorithm [47][7], which we speculate is not so well optimised for simple tree-like queries or long paths that may give rise to multiple lonely variables at the end.

C-Ring uses the least space among all the indexes but, like Ring, offers quite consistent times, being roughly a constant factor slower than Ring for each pattern. Compared with Qdag, the closest index in terms of space, C-Ring is faster in 7 patterns and slower in 10. As Table 1 shows, however, C-Ring is much more stable than Qdag.

5.3 Real-world benchmark

In order to test these systems for a realistic workload at scale, we perform experiments evaluating basic graph patterns taken from real-world queries over the full Wikidata graph of $n = 958{,}844{,}164$ triples, which occupies 10.7 GB in plain form and 7.9 GB in packed form. In search of challenging examples, we downloaded queries that gave timeouts from the Wikidata query logs [28], and selected queries with a single basic graph pattern, obtaining 1,315 unique queries. The minimum, mean and maximum number of triple patterns and variables per query were (1, 2.4, 22) and (1, 2.6, 16), respectively. Unlike the queries used previously, this set contains constant subjects and objects, variable predicates, etc. Letting `s`, `p`, `o` denote any constant and `?` any variable, the most common types of triple pattern patterns were (?, p, ?) (51.5%), (?, p, o) (38.3%), (?, ?, ?) (6.7%), (s, ?, ?) (1.2%), (s, p, ?) (1.2%), (?, ?, o) (1.1%), (s, ?, o) (0.04%).

We exclude EmptyHeaded from these results because its index requires 900 GB and cannot be loaded into our main memory, Qdag because the index does not handle triple patterns with constants in arbitrary positions as occur in this benchmark, and Graphflow as we estimate that it would require terabytes of main memory and does not support constants in arbitrary positions.

[7]EmptyHeaded works with the generalised tree decomposition of queries, which allows one to decompose complex queries into cyclic parts connected through a tree, where the tree is evaluated using Yannakakis' algorithm [47].

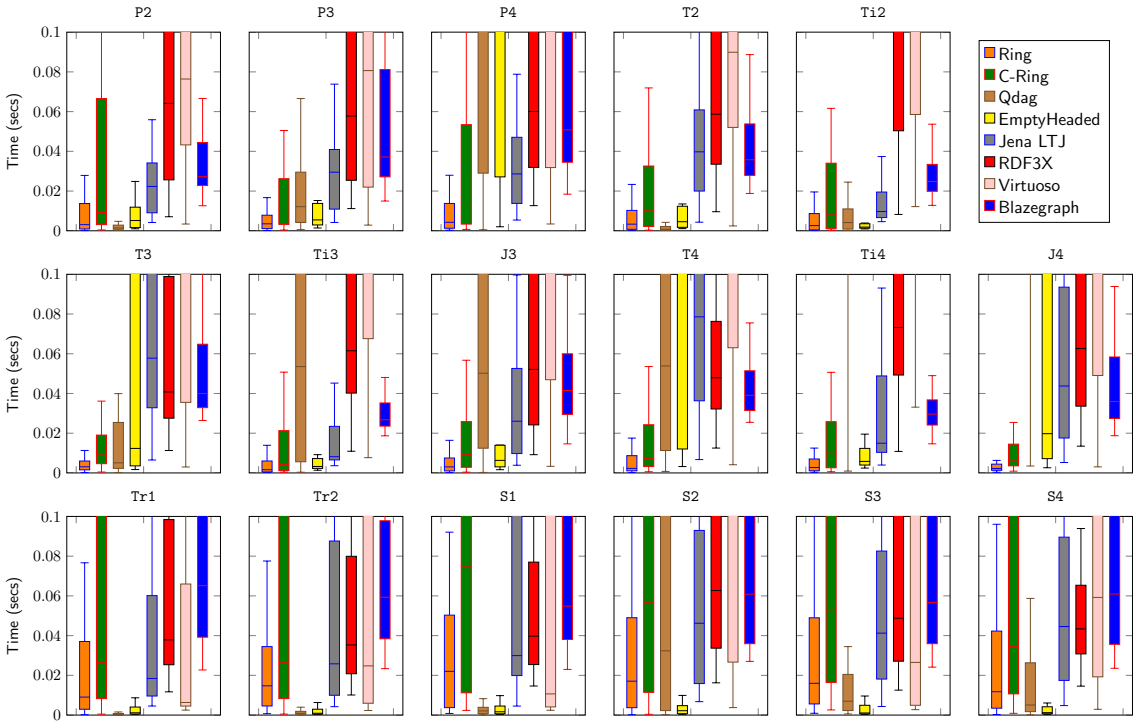


Figure 8: Comparison of query times (in seconds). The boxes span from the 25% to the 75% percentile, with the median marked inside. The lines extend from minima to maxima, removing outliers

The ring index is built over this graph in 2.75 hours (5.8 million triples per minute) using 22 GB of RAM. It occupies 12.4 GB (13.86 bytes per triple) with plain bitvectors. Table 2 shows index space and some statistics on the query times obtained on our benchmark. The two wco-join algorithms (Jena-LTJ, Ring) outperform the alternatives. While Ring is the smallest by a factor of 4.3–12 and has the lowest median, Jena LTJ is twice as fast on average, featuring more stable times in this benchmark. The other systems (all of which are non-wco) are slower than Ring for all the statistics.

6 Rings in higher dimensions

Our ring can be generalised to indexing relations with d attributes, provided we can always choose an index where the bounded variables form a range in the tuples. At each stage of the computation, we maintain the BWT range of the leftmost constant. We can extend the range to include the preceding column in $O(\log U)$ time per intersection step, but extending the range forwards takes us $O(d \log U)$ time, because we have to navigate from the leftmost to the rightmost bounded value for each intersection step. Since there can be $\Theta(d)$ variables in each query tuple, the time that multiplies Q^* (the AGM bound) is now $O(d^2 m \log U)$. Unlike in the case of graphs, which have fixed arity, we cannot handle variables appearing multiple times in the same tuple pattern (i.e., equality selections on attributes of the same relation) without incurring a super-exponential penalty factor on arity in the query time. We provide more details in the supplementary material. Such queries are not typically considered by wco join algorithms in a relational setting. In practice, this penalty may be avoided by only considering equalities that occur in some tuple of the relation instance, or accepting non-wco joins in such cases.

Theorem 6.1. *On a set of n tuples in $[1\,..\,U]^d$, the ring index solves a basic graph pattern query of m tuple patterns with no variables repeated in the same tuple in time $O(Q^* \cdot d^2 m \log U)$, where Q^* is the maximum possible output of such a query on some set of n tuples in $[1\,..\,U]^d$ (the AGM bound). The working space of the query algorithm is $O(v+1)$ words, where v is the number of distinct variables in the query. The size of the ring index is $dn \log_2 U + o(dn \log U)$ bits times the number of orders it has to index.*

The remaining question is how many orders must the ring index so that, independently of the variable elimination order, the constants are always contiguous in the tuple patterns. We have shown that one order suffices for $d = 3$, but we need more for $d \geq 4$.

A classical index (called flat in Figure 2) on d columns needs to store, in principle, all the $d!$ possible orders to support wco algorithms. We refer to these indexes as class W (for Worst-case-optimal). For $d \geq 4$, we can reduce the number of index orders needed through a strategy we call *trie switching*, which allows for variables already bound to be reordered. For example, when indexing quads (s, p, o, g), this avoids storing a trie with the order *gspo* if we have tries for *gsop* and *sgpo* since we can process *gs* with the first index and then switch to the second index (reordering to *sg*) if we require p next. This strategy can be enabled, for example, with inter-trie pointers between the corresponding nodes of *gs* and *sg*. We call TW the class of indexes that support trie switching.

Table 3: Number of orders that must be indexed to support wco algorithms depending on the index capabilities

d	Flat		Cyclic		Ring	
	W	TW	CW	CTW	CBW	CBTW
2	2	2	1	1	1	1
3	6	6	2	2	1	1
4	24	12	6	4	2	2
5	120	30	24	8	5	5
6	720	60	120	[10,12]	10	7
7	5040	140	720	[20,25]	[23,37]	[10,13]
8	40320	280	5040	[35,50]	[79,168]	[18,25]

Trie switching can be used with our indexes as well, without storing any extra space: if we have found a range for some contiguous bounded variables in one index, we can search another index for the same variable values in another desired order using backward search, so long as they are contiguous too. Each such change of index costs $O(d \log U)$ time, which is within the time complexity given in Theorem 6.1. Trie switching can be further combined with tuple circularity (C) and bidirectionality (B), giving rise to various classes of indexes. For example, using trie switching, the cyclic unidirectional index [10] is of class CTW and our ring index is of class CBTW. By calling w, tw, cw, ctw, cbw, and $cbtw$ the number of orders that must be indexed by indexes of the corresponding classes, we can prove various bounds (see supplementary material).

Theorem 6.2. *The following bounds hold, with* $sw(d) = \binom{d}{\lfloor d/2 \rfloor}$*:*

- $w(d) = d!$ *and* $cw(d) = (d-1)!$.
- $\lceil cw(d)/2^{d-2} \rceil \leq cbw(d) \leq cw(d)/2$ *for* $d > 2$.
- $tw(d) = \lceil \frac{d}{2} \rceil \cdot sw(d) = \Theta(2^d d^{1/2})$.
- $\lceil tw(d)/d \rceil \leq ctw(d) \leq tw(d-1)$, *so* $ctw(d) = \Omega(2^d d^{-1/2})$.
- $ctw(d) \leq 2 \cdot sw(\lfloor d/2 \rfloor) \cdot tw(\lceil d/2 \rceil) = O(2^d)$.
- $\lceil ctw(d)/2 \rceil \leq cbtw(d) \leq ctw(d)$, *so* $cbtw(d) = \Theta(ctw(d))$.

Thus, the ring must index between $\Omega(2^d d^{-1/2})$ *and* $O(2^d)$ *orders.*

We ran exhaustive searches to find the exact number of orders that suffice for running wco algorithms in each case, for $d \leq 8$. When the search space was too large, we resorted to approximation algorithms for set cover. Table 3 shows the results, which verified our exact formulas for $w(d)$, $tw(d)$, and $cw(d)$. However, there is a gap between our lower bounds for $ctw(d)$, $cbw(d)$, and $cbtw(d)$, and the approximations we obtained for the larger values of d.

In summary, our support for cyclic bidirectional tuples slashes the number of required orders by an order of magnitude, enabling wco algorithms on dimensions that would be intractable with classical approaches (flat indexes), even using trie switching.

In comparison, Qdag [35] needs to index only one order, but query time is $O(Q^* \cdot 2^d m \log U)$. On the other hand, a cyclic unidirectional index would have query time complexity $O(Q^* \cdot dm \log U)$, by always going backwards, but it would need to index $ctw(d)$ orders, roughly twice as many as our ring index. We can implement such a cyclic unidirectional index with our BWT too, while extending the index of Brisaboa et al. [10] would yield $O(Q^* \cdot d^2 m \log U)$ time because it can move forwards, not backwards.

7 Conclusions

We have introduced the *ring*: an index that regards the triples of a graph database as cyclic and bidirectional, so that it can simulate the 6 triple orders as one. The ring index supports the worst-case-optimal Leapfrog TrieJoin algorithm using almost no extra space on top of the raw triple data, and even in compressed space. Our ring index further offers fast on-the-fly statistics to help query optimisation. Our experiments show that the ring index uses a fraction of the space of traditional indexes while ranking amongst the best in terms of query response times as well.

Regarding future directions, per Section 6, it would be of interest to investigate ring indexes that support higher-arity relations for relational databases, property graphs, etc. An interesting trade-off to explore is between maintaining high-arity relations, which necessitate fewer joins, and decomposing those relations into several lower-arity relations, which necessitate fewer index orders.

Currently our index is read-only, but it could support insertion and deletion of graph edges in $O(\log U \log n)$ time using a compressed representation of string collections based on dynamic wavelet trees [27]. This structure would still use $|G| + o(|G|)$ bits on a graph G (and even compressed space); the price would be an additional penalty factor of $O(\log n)$ in query times. Alternatively, we can trade such a penalty factor for amortised update times by taking the union of results over a small dynamic text index where new triples are added, and a constant amount of increasing static rings for handling space overflows [32]. Various static rings can be merged periodically with the dynamic index to build a bigger ring, which can be done with efficient algorithms to merge BWTs [7].

Our focus has been on indexing for wco joins, where our query planning strategy is currently based on simple techniques. Future work could explore further techniques from the literature, such as tree decompositions for variable ordering [1], low-level caching techniques to reuse intermediate results [24], adaptive plans that use statistics collected during query evaluation [30], hybrid plans that combine wco and non-wco join algorithms for higher-arity relations [19, 30], among others. Our index may further support custom optimisations. For example, a useful statistic would be to find how many different elements are associated with a BWT range, which can be computed, at least in the backward direction, in logarithmic time, by roughly doubling the space [20]. The backward direction also enables faster intersection [21], which makes optimisations using a purely backwards (unidirectional) BWT-based index worth exploring. Supporting further query operators, such as projection, regular path queries, aggregation, etc., would also be of interest, particularly regarding the possibilities of pushing such operators to low-level operations on the index.

More generally, we believe that this paper opens up many interesting lines of research regarding time vs. space trade-offs in the context of worst-case optimal join algorithms.

Source code, scripts, queries, data, instructions for running experiments and omitted proofs are available online [5].

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